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ON THE STABILITY OF A SATELLITE CYLINDRICAL PRECESSION IN AN ELLIPTIC ORBIT

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The stability of motion of a dynamically symmetric satellite with respect to its center of mass in a central Newtonian gravitational field is investigated. The satellite is a solid body whose center of mass moves on an elliptic orbit. The particular case in which the satellite axis of symmetry is normal to the orbit plane (the so-called cylindrical precession [1, 2]) and its absolute angular velocity projection on the axis of symmetry is zero, is examined. Analytical and numerical methods are used. Regions of Liapunov instability and of stability in the first approximation are obtained in the parameter space of the problem (the inertial parameter and the orbit eccentricity). Detailed nonlinear analysis is carried out in the latter, and the formal stability of the satellite cylindrical precession is proved. The question of stability for the majority of intial conditions is also considered [4].

1. Let $OXYZ$ be an orbital system of coordinates whose OX-axis is directed along the transversal to the orbit, the OY-axis lies on the binormal, and the OZ-axis along the radius vector of satellite mass center O. We denote by $Oxyz$ the system of coordinates attached to the satellite whose Oz-axis is directed along the satellite axis of symmetry. The position of the body system of coordinates relative to the orbital one is determined by Euler's angles ψ, ϑ and φ (ψ is the precession angle, ϑ is the nutation angle, and φ is the angle of spin).

The linear dimensions of the satellite are considered to be small in comparison with those of the orbit, hence it is possible [2] to assume that the motion of the satellite relative to its center of mass does not affect the orbit of the center of the mass itself. On these assumptions the satellite motion relative to its center of mass can be defined by canonical differential equations with the Hamilton function of the form [5]

$$H = \frac{p_\psi^2}{2(1+e\cos\nu)^2\sin^2\vartheta} + \frac{p_\vartheta^2}{2(1+e\cos\nu)^2} - p_\psi \operatorname{ctg}\vartheta\cos\psi - \tag{1.1}$$

$$\frac{\alpha\beta(1-e^2)^{3/2}}{(1+e\cos\nu)^2} p_\psi \frac{\cos\vartheta}{\sin^2\vartheta} - p_\vartheta\sin\psi + \alpha\beta(1-e^2)^{3/2}\frac{\cos\psi}{\sin\vartheta} +$$

$$\frac{\alpha^2\beta^2(1-e^2)^3}{2(1+e\cos\nu)^2}\operatorname{ctg}^2\vartheta + \frac{3}{2}(\alpha-1)(1+e\cos\nu)\cos^2\vartheta$$

$$\alpha = \frac{C}{A}\quad (0 \leqslant \alpha \leqslant 2),\quad \beta = \frac{r_0}{\omega_0},\quad \omega_0 = \frac{2\pi}{\tau} \tag{1.2}$$

where A and C are the equatorial and the polar moments of inertia of the satellite, e is the eccentricity of the orbit of satellite center of mass, ν is the true anomaly taken as the independent variable, τ is the period of circulation over the orbit, and r_0 is the projection of the satellite absolute angular velocity on the axis of symmetry which is an integral of motion. The momenta denoted in (1.1) by p_ψ and p_ϑ are canonically conjugate to the variables ψ and ϑ.

The equations of motion that correspond to Hamiltonian (1.1) have the following particular solution [6]:

$$\vartheta_0 = \pi/2, \quad \psi_0 = \pi, \quad p_{\vartheta_0} = 0, \quad p_{\psi_0} = 0 \tag{1.3}$$

This solution correponds to the so-called cylindrical precession of the satellite [1] for which the satellite axis of symmetry is at all times normal to the orbit plane, and the satellite itself rotates around its axis of symmetry at constant angular velocity r_0.

The problem of stability of a satellite cylindrical precession in an elliptical orbit was considered in [7, 8]. The basic results of these investigations concern the analysis of stability of linearized equations of motion and the approximate analysis of nonlinear oscillations of the satellite axis of symmetry. In this paper the problem of stability of the satellite cylindrical precession is considered in a strictly nonlinear formulation, and is restricted to the case when parameter β in the Hamiltonian (1.1) is zero, which corresponds to the translational motion of the satellite in absolute space. Analytical and numerical methods are used, respectively, for small and arbitrary e.

2. Substituting in the Hamiltonian (1.1) the following variables:

$$\vartheta = \pi/2 + q_1, \quad \psi = \pi + q_2, \quad p_\vartheta = p_1, \quad p_\psi = p_2 \tag{2.1}$$

we obtain the formula for the Hamilton function of perturbed motion. It can be readily verified that odd power forms H_m are absent from the series expansion of the Hamiltonian of perturbed motion. We have

$$H = H_2 + H_4 + H_6 + \cdots \tag{2.2}$$

which is obtained by a setting in (1.1) β equal zero. Forms H_2 and H_4 which will be required in the subsequent analysis are

$$H_2 = \frac{1}{2(1+e\cos\nu)^2}(p_1^2 + p_2^2) + p_1 q_2 - q_1 p_2 + \frac{3}{2}(\alpha - 1)(1 + e\cos\nu)\, q_1^2 \tag{2.3}$$

$$H_4 = -\frac{1}{2}(\alpha - 1)(1 + e\cos\nu)\, q_1^4 + \frac{1}{2(1+e\cos\nu)^2} q_1^2 p_2^2 - \frac{1}{3} q_1^3 p_2 + \frac{1}{2} q_1 q_2^2 p_2 - \frac{1}{6} q_2^3 p_1 \tag{2.4}$$

The stability of the satellite cylindrical precession in the case of a circular orbit ($e = 0$) was investigated by numerous authors [1, 5, 9 — 14]. It follows from [5, 9—11, 14] that on a circular orbit for $\beta = 0$ the precession is Liapunov stable for all values of parameter α within the interval $1 < \alpha < 4/3$. Note that when $0 \leqslant \alpha < 1$ and $4/3 < \alpha \leqslant 2$ the considered precession on a circular orbit is Liapunov unstable [9 — 13].

3. Let us first consider the stability on an elliptic orbit in the first approximation. For small eccentricities we shall consider the problem of parametric resonance. For $e = 0$ the canonical substitution of variables $q_i, p_i \to q_i', p_i'$, carried out by formulas

$$\begin{gathered} q_1 = a_1 p_1' - a_2 p_2', \quad q_2 = -a_1\omega_1(1 + b_1) q_1' - a_2\omega_2(1 + b_2) q_2' \\ p_1 = a_1\omega_1 b_1 q_1' + a_2\omega_2 b_2 q_2', \quad p_2 = a_1 b_1 p_1' - a_2 b_2 p_2' \end{gathered} \tag{3.1}$$

$$b_i = \frac{1-\omega_i^2}{1+\omega_i^2}, \quad a_i = \frac{1+\omega_i^2}{\sqrt{\omega_i\,|\,\omega_i^2-1\,|\,(\omega_i^2+3)}}$$

where $\omega_i = \omega_i(\alpha)$ $(i = 1, 2;\ \omega_1 > \omega_2 > 0)$ are the roots of equation

$$\omega^4 - (3\alpha - 1)\,\omega^2 + (4 - 3\alpha) = 0 \qquad (3.2)$$

reduces form H_2 to

$$H_2 = {}^1/_2\omega_1(q_1'^2 + p_1'^2) - {}^1/_2\omega_2(q_2'^2 + p_2'^2) \qquad (3.3)$$

For $0 < e \leqslant 1$ instability is possible owing to the occurence of parametric resonance. With the use of the Krein-Gel'fand-Lidskii theorem [15] we find that for small e instability is only possible in the considered problem when parameter α is such that at least one of the quantities $2\omega_1, 2\omega_2$ or $\omega_1 - \omega_2$ is an integer.

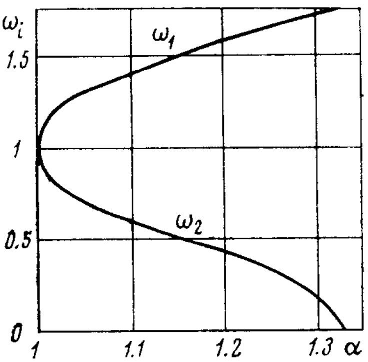

Fig. 1

The dependence of frequencies ω_1 and ω_2 on α is shown in Fig. 1. Using (3.2) it can be shown that for small e and $1 < \alpha < {}^4/_3$ instability is possible in this problem only in the following three cases:

$$\omega_1 = {}^3/_2, \quad \omega_2 = {}^1/_2, \quad \omega_1 - \omega_2 = 1 \qquad (3.4)$$

The corresponding values of parameter α are $\alpha_1 = 1.1603$, $\alpha_2 = 1.1500$ and $\alpha_3 = 1.1547$

When $e \neq 0$ the instability regions issue from points α_i $(i = 1,2,3)$ in the e, α-plane. Equations of the instability region boundaries can be analytically determined if e is small. It appears that the instability region which corresponds to the first of resonances (3.4) is present only in the third approximation by e, while that which corresponds to the second and third resonances (3.4) is present even in the first approximation by e Calculations have shown that the equations of related instability regions are of the form

$$\alpha = \alpha_2 \pm e\cdot 0.015 + O(e^2), \quad \alpha = \alpha_3 \pm e \cdot 0.211 + O(e^2)$$

The investigation of stability in the case of considerable eccentricity necessitates the use of a computer. This requires, first, the determination of the fundamental system of solutions of the linear system of differential equations that correspond to Hamiltonian (2.3) with constant e and α, when $\nu = 2\pi$, followed by the calculation of coefficients of the characteristic equation

$$\rho^4 - a_1\rho^3 + a_2\rho^2 - a_1\rho + 1 = 0$$

If the inequalities

$$-2 < a_2 < 6, \quad 4(a_2 - 2) < a_1^2 < {}^1/_4(a_2 + 2)^2$$

are also satisfied, the considered values of parameters e and α belong to the stability region in the first approximation [16]. The first approximation stability regions (hatched) and regions of Liapunov instability of the satellite cylindrical precession in the e, α-plane are shown in Fig. 2. The illustrated results of stability investigations are a refinement of those presented in [8] for the case of small e. It is rather difficult to derive numerically the instability region issuing from point α_1, owing to the smallness of eccentricity. That region is shown schematically in Fig. 2.

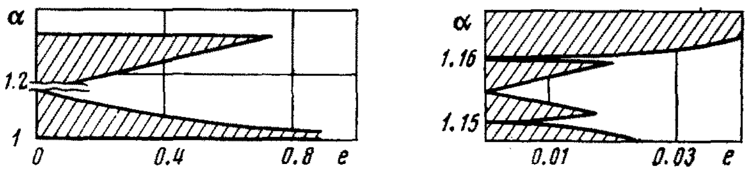

Fig. 2

4. It is necessary to resort to nonlinear analysis for complete investigation of the stability region of the linear system of differential equations that correspond to the Hamiltonian (2.3). When parameters e and α are contained in the region where the necessary conditions of stability are satisfied, the Hamiltonian (2.3) can be reduced to the form

$$H_2 = {}^1/_2\lambda_1 (q_1''^2 + p_1''^2) + {}^1/_2\lambda_2 (q_2''^2 + p_2''^2) \tag{4.1}$$

by the real canonical 2π-periodic with respect to ν change of variables $q_i, p_i \to q_i''$, p_i''. This transformation must be carried out on a computer when the small parameter is absent. The algorithm of related calculations is presented in [17]. Instability is possible for values of parameters e and α from the hatched region in Fig. 2. In the first instance it can occur for such values of parameters e and α at which the third or fourth order resonance

$$n_1\lambda_1 + n_2\lambda_2 = N \tag{4.2}$$

is present. In this formula n_i and N are integers and $|n_1| + |n_2| = 3$ or 4.

Conditions of instability of Hamiltonian systems with resonances (4.2) were obtained in [18].

In the considered problem third order resonances do not induce instability, since in expansion (2.2) of the Hamiltonian of perturbed motion the third power form with respect to q_i and p_i is absent. Fourth order resonances require detailed investigation.

Let there be a single fourth order resonance, i.e. formula (4.2) is satisfied only for one pair of integers n_1 and n_2 the sum of whose absolute values is equal four. If the signs of n_1 and n_2 are different, then, according to [3], formal stability or instability exists for any arbitrarily high finite nonlinear approximation with respect to q_i, p_i. When n_1 and n_2 are of the same sign, for instance $n_i \geqslant 0$, then, by a nonlinear 2π-periodic with respect to ν real canonical substitution of variables q_i'', $p_i'' \to q_i^*$, p_i^*, it is possible to reduce the Hamiltonian to the form [18]

$$\begin{aligned} H &= \lambda_1 r_1 + \lambda_2 r_2 + G(r_1, r_2) + r_1^{1/2 n_1} r_2^{1/2 n_2} \times \\ &\quad [\beta_{n_1, n_2} \sin(n_1\varphi_1 + n_2\varphi_2 - N\nu) + \gamma_{n_1, n_2} \cos(n_1\varphi_1 + n_2\varphi_2 - \\ &\quad N\nu)] + O((r_1 + r_2)^{5/2}) \\ G(r_1, r_2) &= l_{20} r_1^2 + l_{11} r_1 r_2 + l_{02} r_2^2 \\ q_i^* &= \sqrt{2r_i} \sin\varphi_i, \quad p_i^* = \sqrt{2r_i} \cos\varphi_i \end{aligned} \tag{4.3}$$

(l_{ij}, β_{n_1, n_2} and γ_{n_1, n_2} are real numbers).

When the small parameter is absent, the coefficients of expansion (4.3) must be determined on a computer.

If the inequality
$$|G(n_1, n_2)| < n_1^{1/2 n_1} n_2^{1/2 n_2} \sqrt{\beta_{n_1, n_2}^2 + \gamma_{n_1, n_2}^2} \tag{4.4}$$
(for $n_i = 0$, $n_i^{1/2 n_i} = 1$) is satisfied, the considered motion is Liapunov unstable. For the opposite sign in the inequality (4.4) stability is present when at least up to and including fourth order forms are taken into consideration in the expansion of the Hamiltonian. Finally, when the fourth order form with respect to q_i^*, p_i^*
$$F = G(r_1, r_2) + r_1^{1/2 n_1} r_2^{1/2 n_2} [\beta_{n_1, n_2} \sin(n_1 \varphi_1 + n_2 \varphi_2) + \gamma_{n_1, n_2} \cos(n_1 \varphi_1 + n_2 \varphi_2)]$$
is of fixed sign, the motion is formally stable.

If resonances up to and including fourth order are absent, β_{n_1, n_2} and γ_{n_1, n_2} in Hamiltonian (4.3) are identically zero. In that case, under condition of fixed sign of the quadratic form $G(r_1, r_2)$ the motion in region $r_i \geqslant 0$ is formally stable [19]. If, however, the inequality
$$D = l_{11}^2 - 4 l_{20} l_{02} \neq 0 \tag{4.5}$$
is satisfied, the unperturbed motion is stable (in the sense of Lebesgue measure) for the majority of initial conditions [4].

5. The stability and instability conditions formulated in Sect. 4 were applied in numerical and analytical investigation of the stability of satellite cylindrical precession. First, let us consider the case of small eccentricity. First approximation computations show the existence of 13 fourth order resonance curves in the stability region. Ten of these curves correspond to integers n_i of identical sign, which appear in the resonance relationship (4.2). For $e = 0$ the resonance curves issue in the e, α-plane from points $\alpha^{(0)}$ specified in Table 1.

Table 1

№	Resonance	$\alpha^{(0)}$	$\alpha^{(2)}$	№	Resonance	$\alpha^{(0)}$	$\alpha^{(2)}$
1	$4\lambda_2 = -3$	1.0408	−0.0746	8	$3\lambda_1 + \lambda_2 = 4$	1.1574	8.7770
2	$\lambda_1 + 3\lambda_2 = -1$	1.0409	−0.0751	9	$3\lambda_1 - \lambda_2 = 5$	1.1671	2.2475
3	$2\lambda_1 + 2\lambda_2 = 1$	1.0410	−0.0755	10	$4\lambda_2 = -1$	1.2757	0.1032
4	$3\lambda_1 + \lambda_2 = 3$	1.0411	−0.0760	11	$\lambda_1 + 3\lambda_2 = 1$	1.2859	0.0990
5	$4\lambda_1 = 5$	1.0412	−0.0764	12	$2\lambda_1 + 2\lambda_2 = 3$	1.2986	0.0949
6	$\lambda_1 - 3\lambda_2 = 3$	1.1459	−2.2114	13	$\lambda_1 - 3\lambda_2 = 2$	1.3243	0.0028
7	$\lambda_1 + 3\lambda_2 = 0$	1.1523	−8.6744				

When e is small, the equations of resonance curves are of the form
$$\alpha = \alpha^{(0)} + e^2 \alpha^{(2)} + \ldots \tag{5.1}$$
where $\alpha^{(2)}$ is defined by formula
$$\alpha^{(2)} = \frac{n_1 \lambda_1^{(2)} + n_2 \lambda_2^{(2)}}{n_2 d\omega_2/d\alpha - n_1 d\omega_1/d\alpha} \tag{5.2}$$
in which $\alpha = \alpha^{(0)}$ is to be set. The quantities $\lambda_i^{(2)}$ in formula (5.2) are the coefficients at e^2 in the expansion of λ_i that appear in the Hamiltonians (4.1) and (4.3) in powers of the eccentricity

$$\lambda_1 = \omega_1 + e\lambda_1^{(1)} + e^2\lambda_1^{(2)} + \ldots, \quad \lambda_2 = -\omega_2 + e\lambda_2^{(1)} + e^2\lambda_2^{(2)} + \ldots$$

Analysis shows that $\lambda_1^{(1)} = \lambda_2^{(1)} = 0$. The quantities $\lambda_i^{(2)}$ may be represented as functions of ω_1 and ω_2, and

$$\lambda_1^{(2)}(\omega_1, \omega_2) = \frac{(\omega_1^2 - 1)(3\omega_1^6 + 60\omega_1^4 + 17\omega_1^2 - 8)}{4\omega_1(\omega_1^2 + 3)^2(4\omega_1^2 - 1)} - \frac{(1 - \omega_1^2\omega_2^2)(3\omega_1^4 + \omega_2^4 + 6\omega_1^2 - 4\omega_2^2 - 5)}{\omega_1(9 - \omega_1^2\omega_2^2)(\omega_1^4 + \omega_2^2 - 5)}, \quad \lambda_2^{(2)} = \lambda_1^{(2)}(-\omega_2, \omega_1)$$

The quantities $\alpha^{(2)}$ in Eqs. (5.1) of resonance curves are adduced in the table.

For $e = 0$ the coefficients l_{ij} of function (4.3) were obtained analytically. After some rather cumbersome computations they can be written as follows:

$$l_{20} = -\frac{(1 - \omega_1^2)^2}{4(3 + \omega_1^2)^2}, \quad l_{11} = \frac{2(\omega_1^2 + \omega_2^2 - 6)}{\omega_1\omega_2(\omega_1^2 + \omega_2^2 + 6)}$$

$$l_{02} = -\frac{(1 - \omega_2^2)^2}{4(3 + \omega_2^2)^2}$$

Using Eq. (3.2) we find that in the interval $1 < \alpha < {}^4/_3$ all l_{ij} are negative. This implies that for $e = 0$ the quadratic form $G(r_1, r_2)$, defined in (4.3) has a fixed sign in the region $r_1 \geqslant 0$ and $r_2 \geqslant 0$. Hence for fairly small eccentricities and in the absence of fourth order resonances the satellite cylindrical precession is formally stable.

Let us further show that for small e and absence of fourth order resonances the satellite cylindrical precession is stable under the majority of initial conditions. To do this it is necessary to ascertain that inequalities (4.5) are satisfied when $e = 0$. Simple

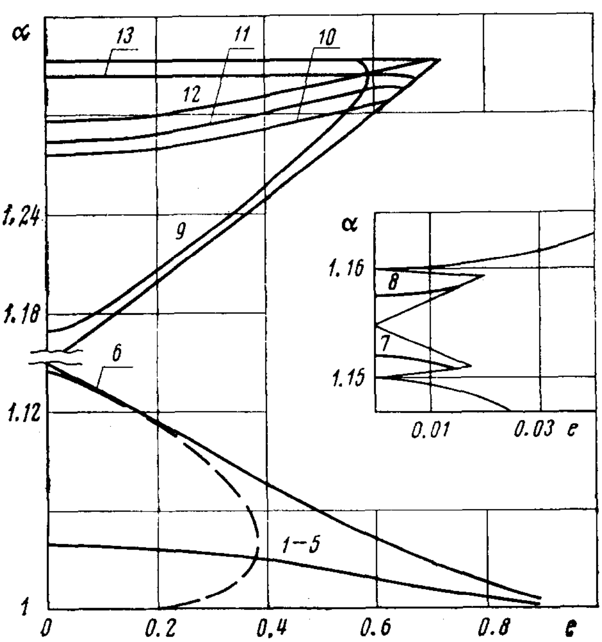

Fig. 3

computations show that

$$D = l_{11}^2 - 4l_{20}l_{02} = \frac{144u^3 + 95u^2 + 18u - 1}{4(9u-1)^2}, \quad u = \omega_1^{-2}\omega_2^{-2}$$

With the use of Eq. (3.2) we obtain that for $1 < \alpha < 4/3$ the quantity $u > 1$. This clearly implies that $D > 0$ and, consequently, condition (4.5) of stability is satisfied for the majority of initial conditions when $e = 0$.

6. Numerical computations were used in the investigation of stability for arbitrary values of parameters α and e contained in the stability region of the first approximation. Curves of fourth order resonances in the e, α -plane are shown in Fig. 3, where where they are denoted by numbers conforming to those in Table 1. They issue from points $\alpha^{(0)}$ in a direction normal to the α-axis. Since curves *1—5* lie very close to each other, they are represented by a single line. Calculations show that in the first approximation the satellite cylindrical precession is formally stable throughout the stability region irrespective of the presence or absence of resonance (note that stability at resonance curve intersections was not investigated).

The test of condition (4.5) shows that stability under the majority of initial conditions is present almost everywhere in the stability region in the first approximation, except possibly at the fourth order resonances curves and at curve $D = 0$ shown in Fig. 3 by a dash line.

The described investigations make it possible to formulate the basic conclusion of this paper in the form of the following statement. The cylindrical presession of a satellite which does not spin around its axis is formally stable for any values of parameters e and α that belong to hatched stability region (Fig. 2) in the first approximation and do not coincide with the intersection points of fourth order resonance curves; if, furthermore the parameters belonging to fourth order resonance curves and curve $D = 0$ shown in Fig. 3 by the dash line, are excluded, the investigated satellite precession is stable (in the sense of Lebesgue measure) under the majority of initial conditions.

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ASYMPTOTIC STABILITY AND SMOOTH EQUIVALENCE OF ORDINARY EQUATIONS

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Under certain specified conditions the asymptotic stability is a coarse property [1], (i.e. addition of fairly smooth functions to the right-hand sides of equations does not disturb the asymptotic stability). It is shown below that in this case the unperturbed system is coarse in a more general sense, namely, any smooth system acted upon by fairly small smooth perturbations, can be returned to its unperturbed state by a smooth reversible transformation. The value and order of the perturbations and the domain of existence of the transformation are all estimated explicitly. The condition required for the above assertion to hold, is that of the existence of a Liapunov function admitting, together with its derivative, specified estimates. This requirement holds, in particular, in the case when the right-hand sides of the unperturbed system are homogeneous functions, the